The KGBR Viewpoint-Lighting Ambiguity and its Resolution by Generic Constraints.

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Abstract

We describe a novel viewpoint-lighting ambiguity which we call the KGBR. This ambiguity assumes orthographic projection or an affine camera, and uses Lambertian reflectance functions including cast/attached shadows and multiple light sources. A KGBR transform alters the geometry (by a three-dimensional affine transformation) and albedo properties of objects. If two objects are related by a KGBR transform then for any viewpoint and lighting of the first object there exists a corresponding viewpoint and lighting of the second object so that the images are identical up to an affine transformation. The Generalized Bas Relief (GBR) ambiguity [1] is obtained as a special case of the KGBR. We describe generic viewpoint and lighting assumptions [5] and show that either, or both, resolve this ambiguity by biasing towards objects with planar geometry.

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1 Introduction

From prehistoric times artists have carved portraits in bas relief (see the Assyrian art in the British Museum). In such portraits the relative depth of the object is compressed (mathematically there is a transformation $z \mapsto \lambda z$ where z is the depth and $\lambda < 1$ is the compression factor). The shading and shadows of the resulting sculptures, however, appear to be very realistic. Similar bas relief sculptures occur at the Yale University campus (P. Belhumeur – private communication).

Recent work by Belhumeur, Kriegman and Yuille [1] has shown that there is a generalized bas relief (GBR) ambiguity, see figure (1), in the equations for shading and shadows (assuming a Lambertian reflectance model with cast and attached shadows). The GBR includes the standard bas relief ambiguity as a special case. (This ambiguity is of practical consequence in photometric stereo, see [9]).

The first goal of this paper is to demonstrate a novel viewpoint-lighting ambiguity which we call the KGBR. (The name is obtained by adding "K" to "GBR" where "K" stands for the German mathematician Felix Klein who pioneered the study of group theory for geometry). The KGBR acts on an object by a three-dimensional affine transformation on the object's geometry combined with a transformation on the object's albedo. If two objects are related by a KGBR transform then for any viewpoint and lighting of the first object there exists a corresponding viewpoint and lighting of the second object so that the images are identical up to an affine transformation. We prove that the Generalized Bas Relief (GBR) ambiguity is a special case of the KGBR. It is obtained by imposing the additional constraint that there exists a special viewpoint from which the two objects appear identical.

The KGBR gives a direct link between shadow/shading ambiguities, such as the GBR, and ambiguities in surface geometry which arise in structure from motion for objects composed of point-like features [7],[8]. Indeed, for any linear ambiguity which arises in multiple view geometry [6] for point-like features, we can obtain an identical ambiguity for shading and shadowed surfaces by using the KGBR.

The second goal of this paper is to investigate the effect of the generic viewpoint/lighting assumption [5],[3] on the KGBR ambiguity. The intuition behind this assumption is that human observers tend to avoid interpretations of the data which correspond to "accidental", or unusual, lighting and/or viewpoint conditions. This assumption is formulated mathematically by treating viewpoint and light as "nuisance variables" which should be integrated over [5].

Our results prove that the generic viewpoint/lighting assumptions causes a bias towards planar objects. In other words, an observer would prefer to interpret an image as if it were a flat painting. These results are consistent with earlier results by Weinshall and Werman [11] who analyzed the stability of the perception of three-dimensional shapes (but who did not consider shading or shadows). The results also extend our previous work [10] where we applied the generic lighting assumption to disambiguate the GBR.

In section (2) we define the KGBR and prove that it preserves the shading and shadow properties of surfaces as the lighting changes. Section (3) proves that the KGBR gives a joint viewpoint-lighting ambiguity. In section (4) we show that the GBR ambiguity can be obtained as a special case of the KGBR. Section (5) gives an example of the KGBR applied to faces. In section (6) we prove that generic assumptions on lighting and viewpoint resolve the KGBR ambiguity.

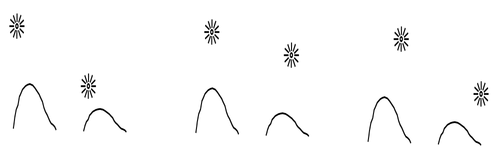

Figure 1: If the lighting conditions are unknown, then it is impossible to distinguish between two objects related by a GBR (generalized bas relief) ambiguity. For any image of the first object, under one illumination condition, we can always find a corresponding illumination condition which makes the second object appear identical (i.e., generate an identical image). We show two objects under three different, but corresponding, lighting conditions.

2 The KGBR

In this section, we define the KGBR transforms on the surface, albedo, and light source directions. We show that this will preserve the shading and shadows *on the surface*. In the following section, we see what this implies about the projection of the object to different viewpoints.

Suppose we have a surface $\vec{r} = \vec{r}(u,v)$ where (u,v) are coordinates on the surface. The surface normal is denoted by $\vec{n}(u,v)$. Let $a(u,v)$ be the surface albedo and let $\vec{s}$ denote the lighting. (In this paper we will assume a single light source for simplicity, but the results generalize directly to multiple light sources).

Definition 1 *A KGBR transform takes* $\vec{r}(u,v), \vec{n}(u,v), a(u,v), \vec{s}$ *to* $\vec{\hat{r}}(u,v), \vec{\hat{n}}(u,v), \hat{a}(u,v), \vec{\hat{s}}$ *where:*

$$\vec{\hat{r}}(u,v) = \mathbf{K}\vec{r}(u,v), \quad \hat{a}(u,v) = a(u,v) det\mathbf{K}|\mathbf{K}^{-1,T}\vec{n}(u,v)| \tag{1}$$

$$\vec{\hat{n}}(u,v) = \frac{\mathbf{K}^{-1,T}\vec{n}(u,v)}{|\mathbf{K}^{-1,T}\vec{n}(u,v)|}, \quad \vec{\hat{s}} = \frac{1}{det\mathbf{K}}\mathbf{K}\vec{s}. \tag{2}$$

The matrix $\mathbf{K}$ of the KGBR can take any form. It gives an affine transformation on the three-dimensional surface which can involve squashing, skewing, or rotating the surface (or any combination of these operations). The form of $\vec{\hat{n}}(u,v)$ in equation (2) is derived directly from the transformation on the surface shape $\vec{r}(u,v)$. (To see this, recall that $\vec{\hat{n}}(u,v)$ must be orthogonal to the surface tangent vectors $\vec{\hat{r}}_u(u,v)$ and $\vec{\hat{r}}_u(u,v)$ which transform by the KGBR.)

The factor $det\mathbf{K}$ in the transformation of the albedo ensures that the albedo remains finite in the limit that the matrix $\mathbf{K}$ becomes non-invertible.

Theorem 1 *If two objects and their lighting are related by a KGBR then their shading and shadows are preserved in the surface coordinates* (u,v).

Proof. *It follows directly from equation (2) that the shadows on the surface are preserved by KGBR transforms (i.e. if point* (u,v) *is in shadow on surface* $\vec{r}(u,v)$ *with light source* $\vec{s}$*, then it is also in shadow on surface* $\vec{\hat{r}}(u,v)$ *under light source* $\vec{\hat{s}}$*). From equation (2), we obtain:*

$$\max\{\hat{a}(u,v)\vec{\hat{n}}(u,v)\cdot\vec{\hat{s}}, 0\} = \max\{a(u,v)\vec{n}(u,v)\cdot\vec{s}, 0\}, \tag{3}$$

and so the shading is also preserved.

3 Viewpoint Projections

The previous section has established that shading and shadow properties at points (u,v) on a surface are preserved by a KGBR transformation. We now determine what happens as we project the surface onto an image plane.

Definition 2 *A projection is specified by two vectors* $\vec{c}_1, \vec{c}_2$*. For orthographic projection these vectors are constrained to be orthogonal unit vectors. For the affine camera there are no constraints. The viewpoint direction for orthographic projection is defined to be* $\vec{v}_1 = \vec{c}_1 \times \vec{c}_2$*. A point* (u,v) *on the surface* $\vec{r}(u,v)$ *is projected to points* $p_1(u,v), p_2(u,v)$ *in the image plane by*

$$p_1(u,v) = \vec{c}_1 \cdot \vec{r}(u,v), \quad p_2(u,v) = \vec{c}_2 \cdot \vec{r}(u,v). \tag{4}$$

We now show that there is a viewpoint-lighting ambiguity, see figure (2). Our results are given in Theorems 2, 3 for

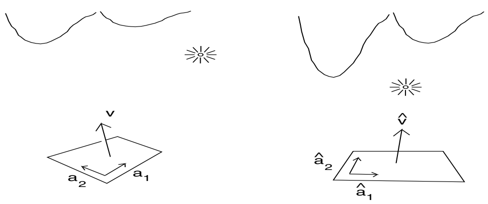

Figure 2: The joint viewpoint-lighting ambiguity. If two objects are related by a KGBR then for any view of one there is a corresponding view of the other which is identical (for the affine camera) or identical up to an affine warp (for orthographic projection). The lighting is also transformed by the cooresponding KGBR.

the affine camera and for orthographic projection respectively. We emphasize that our results only apply to the "visible" portions of the objects because as the observer's viewpoint changes there will be points on the object which are no longer visible (our result does not apply to such points).

Theorem 2. *If two objects' geometry and albedo are related by a KGBR transform* $\mathbf{K}$, *then for any viewpoint and illumination condition of one object there exists a viewpoint and illumination condition for the second object such that the images (of the visible portions) of both objects are identical. The viewpoint projections are given by the affine camera with* $\vec{\hat{c}}_1 = \mathbf{K}^{-1,T}\vec{c}_1$ *and* $\vec{\hat{c}}_2 = \mathbf{K}^{-1,T}\vec{c}_2$.

Proof. *For the affine camera, we use the projection equation (4) to compare the projection of* $\vec{r}(u,v)$ *and* $\vec{\hat{r}}(u,v)$. *By combining equations (2,4) we can express the projection of* $\vec{r}(u,v)$ *by* $p_1(u,v) = \vec{c}_1 \cdot \mathbf{K}^{-1}\vec{\hat{r}}(u,v)$ *and* $p_2(u,v) = \vec{c}_2 \cdot \mathbf{K}^{-1}\vec{\hat{r}}(u,v)$. *This is identical to the projection of* $\vec{\hat{r}}(u,v)$ *using* $\vec{\hat{c}}_1 = \mathbf{K}^{-1,T}\vec{c}_1$ *and* $\vec{\hat{c}}_2 = \mathbf{K}^{-1,T}\vec{c}_2$.

We now extend our result to the more practical case of orthographic projection (it is also straightforward to deal with scaled orthographic). In this case, the projections vectors $\vec{c}_1, \vec{c}_2$ are required to be orthogonal unit vectors. We cannot apply Theorem 2 and set $\vec{\hat{c}}_1 = \mathbf{K}^{-1,T}\vec{c}_1$ and $\vec{\hat{c}}_2 = \mathbf{K}^{-1,T}\vec{c}_2$ because this transformation does not ensure that the vectors $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ are orthogonal unit vectors (except for the uninteresting special case where $\mathbf{K}$ is a rotation matrix).

Instead we loosen our notion of image similarity. Following Werman and Weinshall [12] we consider an affine invariant measure on the set of images.

Definition 3 *Two images* I *and* $\hat{I}$ *are identical up to an affine transformation* $\mathbf{A}$ *provided we can find matrix coefficients* $A_{11}, A_{12}, A_{21}, A_{22}$ *such that the following condition is satisfied:*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix}, \qquad (5)$$

so that $I(p_1,p_2) = \hat{I}(\hat{p}_1,\hat{p}_2)$ *for all* (p_1,p_2).

Theorem 3 *If two objects are related by a KGBR transform* $\mathbf{K}$, *then for any viewpoint and illumination condition of one object there exists a viewpoint and illumination condition for the second object such that the images (of the visible portions) of both objects are identical up to an affine transformation. The projection is orthographic and the viewpoints are related by* $\vec{\hat{v}} = \mathbf{K}\vec{v}/|\mathbf{K}\vec{v}|$.

Proof. *To ensure that projections* $\vec{c}_1, \vec{c}_2$ *and* $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ *of* $\vec{r}(u,v)$ *and* $\vec{\hat{r}}(u,v)$ *are identical up to an affine transformation, see equation (5), requires finding orthogonal unit vectors* $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ *and coefficients* $A_{11}, A_{12}, A_{21}, A_{22}$ *such that:*

$$\vec{\hat{c}}_1 = A_{11}\mathbf{K}^{-1,T}\vec{c}_1 + A_{12}\mathbf{K}^{-1,T}\vec{c}_2,$$
$$\vec{\hat{c}}_2 = A_{21}\mathbf{K}^{-1,T}\vec{c}_1 + A_{22}\mathbf{K}^{-1,T}\vec{c}_2, \qquad (6)$$

where $\vec{c}_1, \vec{c}_2$ *are orthogonal unit vectors. These conditions can always be satisfied provided* $\mathbf{K}$ *is invertible because the vectors* $\mathbf{K}^{-1,T}\vec{c}_1$ $\mathbf{K}^{-1,T}\vec{c}_2$ *span a two-dimensional space and we can select* $A_{11}, A_{12}, A_{21}, A_{22}$ *to ensure that* $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ *are orthogonal (by the Gram-Schmidt orthogonalization process).*

To get more geometrical understanding of equation (6), recall that any orthographic projection corresponds to linear projection onto a plane. The normals to this plane is the viewpoint direction, see Definition 2, and is given by $\vec{v} = \vec{c}_1 \times \vec{c}_2$ *and* $\vec{\hat{v}} = \vec{\hat{c}}_1 \times \vec{\hat{c}}_2$ *respectively. Hence* $\vec{\hat{v}} \propto \mathbf{K}^{-1,T}\vec{c}_1 \times \mathbf{K}^{-1,T}\vec{c}_2$. *It follows directly (using* $\vec{c}_1 \cdot \vec{v}_1 = \vec{c}_2 \cdot \vec{v}_2 = 0$*) that* $\vec{\hat{v}} = \frac{\mathbf{K}\vec{v}}{|\mathbf{K}\vec{v}|}$.

We can also determine the form of the affine transformation $\mathbf{A}$. The results are given by the theorem.

Theorem 4 *The affine transformation* $\mathbf{A}$ *is preserved by the transformations* $\mathbf{A} \mapsto \mathbf{\Psi}\mathbf{A}\mathbf{\Phi}^T$, *where* Ψ *and* Φ *are rotation matrices corresponding to the choice of coordinate systems in the viewing planes. The remaining portions of* $\mathbf{A}$ *are specified by the relations* $\det \mathbf{A} = det\mathbf{K}/|\mathbf{K}\vec{v}|$ *and* $Trace\{\mathbf{A}\mathbf{A}^T\} = Trace\{\mathbf{K}\mathbf{K}^T\} - |\mathbf{K}^T\vec{\hat{v}}|^2$.

Proof. *Let the affine transform be* $\mathbf{A}$. *Then we can determine it in terms of* $\mathbf{K}$ *by the following procedure. Firstly, the requirement that* $\vec{c}_1, \vec{c}_2$ *and* $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ *are pairs of orthogonal unit vectors means that* $det\mathbf{A} = det\mathbf{K}/|\mathbf{K}\vec{v}|$. *Secondly, we have the freedom to rotate the axes* $\vec{c}_1, \vec{c}_2$ *by a rotation* $\mathbf{\Phi}$ *in the first viewing plane and similarly rotate* $\vec{\hat{c}}_1, \vec{\hat{c}}_2$ *by a similar rotation* $\mathbf{\Psi}$ *in the second viewing plane. Therefore we have the freedom to send* $\mathbf{A} \mapsto \mathbf{\Psi}\mathbf{A}\mathbf{\Phi}^T$. *Finally we see that this implies that* $Trace\{\mathbf{A}\mathbf{A}^T\}$ *is constant. By substitution, we obtain that* $Trace\{\mathbf{A}\mathbf{A}^T\} = Trace\{\mathbf{K}\mathbf{K}^T\} - |\mathbf{K}^T\vec{\hat{v}}|^2$. *Thus two of the four degrees of freedom of* $\mathbf{A}$ *are determined uniquely by* $\mathbf{K}$ *and the other two degrees of freedom correspond to the arbitrary choice of coordinate axes in the two viewing planes.*

We have avoided the special case where $\mathbf{K}$ is not invertible. In this degenerate case, equation (2) implies that one object is planar. If the second object is also planar then the results above are easy to prove. If the second object is non-planar, however, then the results no longer hold. This is because all views of the planar object are equivalent to within an affine transformation and therefore only corresponds to a single view of the non-planar object (the front-on view). Thus for almost all views of the non-planar object there is no corresponding view of the planar object. This applies both for the affine camera and for orthographic/scaled-orthographic projections.

4 Relationship to the Generalized Bas Relief (GBR) transformation

The KGBR is a generalization of the Generalized Bas Relief (GBR) transformation to the case where we alter both the lighting *and* the viewpoint. We now show that we can obtain the GBR as a special case of the KGBR.

Theorem 5. *If two objects are related by a KGBR* $\mathbf{K}$ *and there exists a special viewpoint* $\vec{v}^*$ *such that the images of the two objects are identical, then* $\mathbf{K}$ *must be of form* $\mathbf{G}^{-1,T}$ *where* $\mathbf{G}$ *is a GBR.*

Proof. *Let the two objects be* $\vec{r}(u,v)$ *and* $\vec{\hat{r}}(u,v) = \mathbf{K}\vec{r}(u,v)$. *If there exists a special viewpoint (either orthographic or with an affine camera) such that the images are identical then we can find vectors* $\vec{c}_1^*, \vec{c}_2^*$ *such that:*

$$\vec{c}_1^* \cdot \vec{r}(u,v) = \vec{c}_1^* \cdot \mathbf{K}\vec{r}(u,v),\ \forall u,v$$
$$\vec{c}_2^* \cdot \vec{r}(u,v) = \vec{c}_2^* \cdot \mathbf{K}\vec{r}(u,v)\ \forall u,v. \quad (7)$$

This implies that $\mathbf{K}^{-1,T}$ *has two unit eigenvectors* $\vec{c}_1^*, \vec{c}_2^*$ *(unless the surface is degenerate). Hence by a suitable choice of coordinate system we can express* $\mathbf{K}^{-1,T}$ *as:*

$$\mathbf{K}^{-1,T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & \nu & \lambda \end{pmatrix}, \quad (8)$$

which is of GBR form [1]. Conversely, if $\mathbf{K}^{-1,T}$ *has two unit eigenvalues then we can define the projections to be their corresponding eigenvectors.*

5 A Face Example

We now consider a simple example of an object, in this case a face, see figure (3), viewed from approximately front-on. We use the image coordinates (x,y) from the front-on viewpoint as the labels for points on the surface. (I.e. we replace (u,v) by (x,y)).

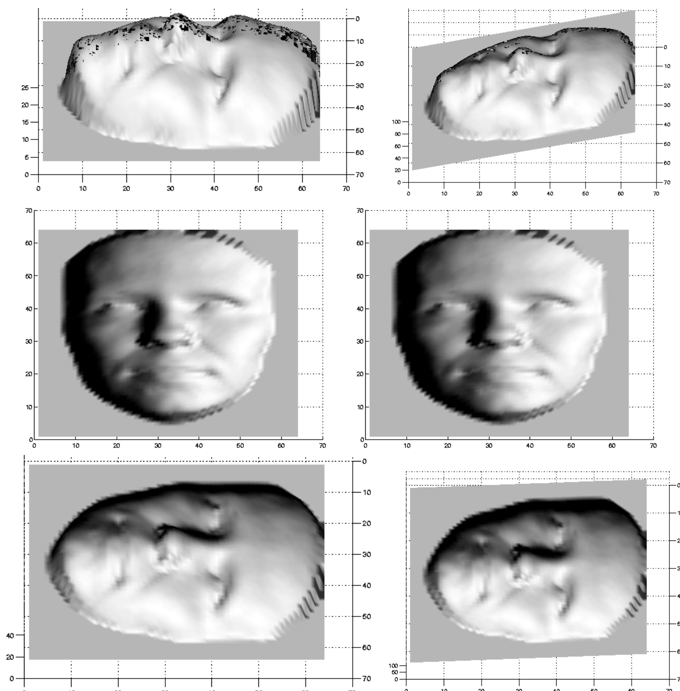

Figure 3: Two Faces related by a KGBR. Top Panel: the faces seen from side-on. Middle Panel: the faces viewed from front-on, so they appear identical. Bottom Panel: the faces viewed from different but corresponding viewpoints (rotating the first object by 15 degrees about the vertical axis) so that the images are similar up to an affine transformation (a scaling in the vertical direction).

Suppose we have one object with $\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$ which is transformed by a GBR to $\vec{\hat{r}}(x,y) = x\vec{i} + y\vec{j} + (\lambda f(x,y) + \mu x + \nu y)\vec{k}$. Now suppose we rotate the viewpoint about the $\vec{i}$ axis. This gives projection vectors $\vec{c}_1 = \vec{i}$ and $\vec{c}_2 = \cos\psi\vec{j} + \sin\psi\vec{k}$ for the first object. We get corresponding projection vectors $\vec{\hat{c}}_1 = \vec{i}$ and $\vec{\hat{c}}_2 = \cos\hat{\psi}\vec{j} + \sin\hat{\psi}\vec{k}$ for the transformed object.

With these projections, we see that points (x,y) on the surfaces will project to points $(x, y\cos\psi + f(x,y)\sin\psi)$ and $(x, y\cos\hat{\psi} + \lambda f(x,y)\sin\hat{\psi} + \mu x\sin\hat{\psi} + \nu y\sin\hat{\psi})$ respectively.

We relate these coordinates by an affine transform $\mathbf{A}$. This gives $A_{11} = 1, A_{12} = 0$, $A_{21} = \mu\sin\psi/\{\sin^2\psi + (\lambda\cos\psi - \nu\sin\psi)^2\}^{1/2}$, $A_{22} = \lambda/\{\sin^2\psi + (\lambda\cos\psi - \nu\sin\psi)^2\}^{1/2}$. Where $\cot\hat{\psi} = \lambda\cot\psi - \nu$.

The images are the same up to the affine transformation $\mathbf{A}$. We observe that for small rotations ψ we get $A_{21} \approx \mu\psi/\lambda$ and $A_{22} \approx 1 + 2\nu\psi/\lambda$.

6 Generic Viewpoint and Lighting Constraints

The generic lighting and viewpoint assumptions were formulated in Bayesian terms by Freeman [5], [3]. It involves treating lighting or viewpoint as a "nuisance" parameter which should be integrated out, see figure (4).

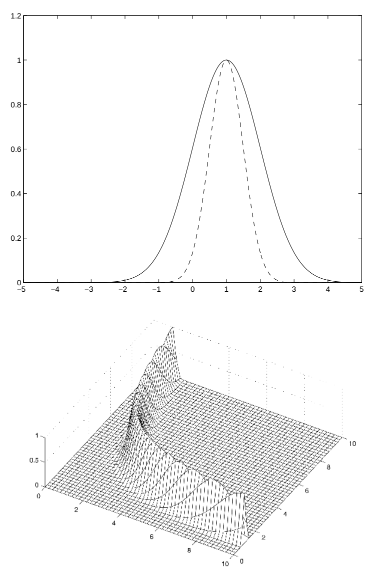

Figure 4: (Top Panel). We plot the probabilities $P(I|b,s)$ (solid line) and $P(I|\hat{b},s)$ (dashed line) as functions of s. (For simplicity I and s are scalars.) The image measurement I is equally likely for object b and $\hat{b}$ assuming that $s=1$, i.e., $P(I|b,s=1) = P(I|\hat{b},s=1)$. However, if we integrate over s we find that $P(I|b) > P(I|\hat{b})$, i.e., there is more evidence for object O if we assume only a uniform prior on s. In other words, the image measurement I is more stable for object b than $\hat{b}$. (Bottom Panel) Brainard and Freeman analyzed the use of the generic assumption for the probability distribution $P(I|s,b) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(I-sb)^2/(2\sigma^2)}$. We plot this as a function of s, b, shown for $I = 10.0$. If the variable s is integrated out then there becomes a unique best estimate for b.

In previous work [10], we analyzed the effect of the generic lighting constraint, as formulated by Freeman [5], on the GBR and showed that it resolved the ambiguity by biasing towards the fronto-parallel plane.

In this section, we generalize this result to the KGBR for viewpoint and lighting. We also demonstrate that there is a bias towards planar objects even if we keep the illumination fixed. This verifies a result by Weinshall and Werman [11] obtained for point/feature sets (i.e. not dealing with shading and shadows) and using a different mathematical framework. Indeed, our result confirms Weinshall and Werman's intuition that an analysis of the generic viewpoint assumption based on Freeman's Bayesian formulation would yield results similar to theirs.

To formulate the generic viewpoint and lighting constraints, we assume that:

$$P(I|\vec{s}, a, \vec{r}, \vec{v}) = F(\vec{s}, a, \vec{r}, \vec{v}, I), \quad (9)$$

where $F(\vec{s}, a, \vec{r}, \vec{v}, I)$ is a function which is invariant to a KGBR. This invariance is guaranteed if the image has Lambertian shading and shadowing, see Theorem 1, modified by *any* noise process to take care of the imaging. For example, we could set $F(\vec{s}, a, \vec{r}, \vec{v}, I) = \mathbf{N}(I - a\vec{n}\cdot\vec{s}; \Sigma)$ where $\mathbf{N}$ is a multivariate Gaussian distribution with mean $I - a\vec{n}\cdot\vec{s}$ and covariance Σ. Alternatively, we could set $F(\vec{s}, a, \vec{r}, \vec{v}, I) = \delta(I - a\vec{n}\cdot\vec{s})$ where $\delta(.)$ is the multivariate Dirac delta function.

We now assume that all lighting $\vec{s}$ is equally likely and all viewpoints $\vec{v}$ are equally likely. This means that the probability that the image I is generated by the object with $a(u,v), \vec{r}(u,v)$ is obtained by integrating $P(I|\vec{s}, a, \vec{r}, \vec{v})$ with respect to $\vec{s}$ and the solid angle $\delta\Omega$. In other words:

$$P(I|a, \vec{r}) = \int d\vec{s} d\Omega F(\vec{s}, a, \vec{r}, \vec{v}, I). \quad (10)$$

To determine how the generic constraints affects the KGBR we must first obtain an expression for how the solid angle $d\Omega$ transforms under a KGBR. The result is stated in the following lemma.

Lemma 1 *If the viewpoint is related by* $\hat{\vec{v}} = \mathbf{K}\vec{v}/|\mathbf{K}\vec{v}|$, *then the solid angles are related by* $d\hat{\Omega} = det\mathbf{K} d\Omega / |\mathbf{K}\vec{v}|^3$.

Proof. *This is a standard result in affine geometry. To obtain it, set* $\vec{v}(\alpha, \beta)$ *to be a unit vector representing viewpoint where* α, β *are coordinates on the unit sphere. Then the solid angle is* $d\Omega = |\vec{v}_\alpha \times \vec{v}_\beta| d\alpha d\beta$. *The solid angle on the transformed surface is* $d\hat{\Omega} = |\hat{\vec{v}}_\alpha \times \hat{\vec{v}}_\beta| d\alpha d\beta$. *By setting* $\hat{\vec{v}} = \mathbf{K}\vec{v}/|\mathbf{K}\vec{v}|$ *we obtain* $|\hat{\vec{v}}_\alpha \times \hat{\vec{v}}_\beta| = det\mathbf{K}/|\mathbf{K}\vec{v}|^3$.

We now obtain our main result on generic viewpoint and lighting.

Theorem 6. *If two objects* $a, \vec{r}$ *and* $\hat{a}, \hat{\vec{r}}$ *are related by a KGBR* $\mathbf{K}$ *(whose eigenvalues are required to all be finite),* *then their likelihood functions, marginalized over viewpoint and lighting, are given by*

$$P(I|a,\vec{r}) = \int d\vec{s}d\Omega F(\vec{s},a,\vec{r},\vec{v},I),$$
$$P(I|\hat{a},\vec{\hat{r}}) = \int d\vec{s}d\Omega \frac{1}{|\mathbf{K}\vec{v}|^3 det\mathbf{K}} F(\vec{s},a,\vec{r},\vec{v},I), \quad (11)$$

and the integrand of $P(I|\hat{a},\vec{\hat{r}})$ *becomes largest as* $\det \mathbf{K} \mapsto 0$ *corresponding to a planar surface. Therefore the most probable interpretation of an image is of a planar surface.*

Proof. *Suppose we have two objects* $a(u,v), \vec{r}(u,v)$ *and* $\hat{a}(u,v), \vec{\hat{r}}(u,v)$ *related by a KGBR* $\mathbf{K}$. *Then we can write the likelihoods for the two objects as:*

$$P(I|a,\vec{r}) = \int d\vec{s}d\Omega F(\vec{s},a,\vec{r},\vec{v},I),$$
$$P(I|\hat{a},\vec{\hat{r}}) = \int d\vec{\hat{s}}d\hat{\Omega} F(\vec{\hat{s}},\hat{a},\vec{\hat{r}},\vec{\hat{v}},I). \quad (12)$$

We now perform a change of variables in the integral for $P(I|\hat{a},\vec{\hat{r}})$ *corresponding to the KGBR. Recalling that the integrand* $F(\vec{s},a,\vec{r},\vec{v},I)$ *is invariant to a KGBR the only change is the Jacobian factors. By equation (2) we see that* $d\vec{\hat{s}} = (1/det\mathbf{K})^2 d\vec{s}$. *And by the previous Lemma we have* $d\hat{\Omega} = det\mathbf{K}d\Omega/|\mathbf{K}\vec{v}|^3$. *This yields:*

$$P(I|\hat{a},\vec{\hat{r}}) = \int d\vec{s}d\Omega \frac{1}{|\mathbf{K}\vec{v}|^3 det\mathbf{K}} F(\vec{s},a,\vec{r},\vec{v},I). \quad (13)$$

The difference between $P(I|a,\vec{r})$ *and* $P(I|\hat{a},\vec{\hat{r}})$ *is the factor* $\frac{1}{|\mathbf{K}\vec{v}|^3 det\mathbf{K}}$ *in the integrand of the later. This factor becomes infinite in the limit as* $\det \mathbf{K} \mapsto 0$ *which corresponds to* $\vec{\hat{r}}$ *becoming planar.*

For any image I and object $a,\vec{r}$ there will be a dominant light source $\vec{s}^*$ and viewpoint $\vec{v}^*$ which maximize $F(\vec{s},a,\vec{r},\vec{v},I)$. The transformation $\mathbf{K}$ who zero eigenvalue lies in the direction of $\vec{v}^*$ is the best one – so the surface gets made fronto-parallel.

We also get a similar result if we keep the illumination fixed and just integrate over viewpoint. This is similar to a result obtained by Weinshall and Werman by different methods.

Theorem 7. *If two objects* $a,\vec{r}$ *and* $\hat{a},\vec{\hat{r}}$ *are related by a KGBR* $\mathbf{K}$ *(whose eigenvalues are required to all be finite), then marginalizing their likelihood functions over viewpoint gives a bias to a planar surface, provided* $F(\vec{s},a,\vec{r},\vec{e},I) \neq 0$ *where* $\vec{e}$ *is the eigenvector of* $\mathbf{K}$ *with zero eigenvalue.*

Proof. *We compare* $P(I|a,\vec{r},\vec{s})$ *with* $P(I|\hat{a},\vec{\hat{r}},\vec{\hat{s}})$. *This requires integrating over viewpoint only. The difference between the two integrands is simply the Jacobian factor* $det\mathbf{K}/|\mathbf{K}\vec{v}|^3$ *for viewpoint. Now let* $\det \mathbf{K} \mapsto 0$ *and let* $\vec{e}$ *be the eigenvector corresponding to the zero eigenvalue of* $\mathbf{K}$. *Then the integral for* $P(I|\hat{a},\vec{\hat{r}},\vec{\hat{s}})$ *will have a contribution from viewpoint* $\vec{v} = \vec{e}$ *which becomes infinitely large as* $det\mathbf{K} \mapsto 0$. *Hence we bias towards a KGBR with* $det\mathbf{K} = 0$ *and the preferred interpretation is a planar surface.*

7 Conclusion

The first goal of this paper was to describe a joint lighting and viewpoint ambiguity which we called the KGBR. We showed that the Generalized Bas Relief (GBR) ambiguity [1] is a special case of the KGBR. Our results assume a Lambertian reflectance model with cast and attached shadows and allows for multiple light sources. The viewpoint is modeled either by an affine camera or by orthographic projection. If orthographic projection is used then the KGBR ambiguity is with respect to an affine invariant image measure [12].

Our second goal was to analyze how the generic lighting and viewpoint assumptions [5] interacted with the KGBR ambiguity. We showed, subject to certain technical constraints, that either or both of these assumptions disambiguated the KGBR ambiguity by biasing towards a planar surface. These results are consistent with earlier results by Weinshall and Werman [11] who analyzed the stability of the perception of three-dimensional shapes (but who did not consider shading or shadows). The results also extend our previous work [10] where we applied the generic lighting assumption to disambiguate the GBR.

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